Elliptic curves in Nemo

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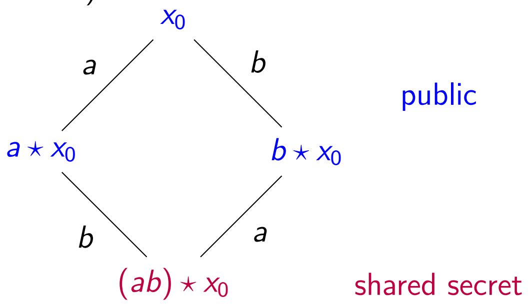
Key exchange from hard homogeneous spaces

Let G be an abelian group acting on a set X with some given point x_0. If the action is

- easy to compute (polynomial time),
- hard to invert (exponential time),

then there is an analogue of the Diffie–Hellman key exchange (Couveignes 2006).



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Goals

- Explain what this means
- Describe the computations needed
- Discuss our EllipticCurves module in Nemo.

Elliptic curves over k

- *Elliptic curves* over a field k are algebraic curves, e.g.

  $$E \ : \ y^2 = x^3 + ax + b.$$

  They have an abelian group structure. The j-invariant

  $$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}$$

  classifies such curves up to isomorphism.
- *Isogenies* are nonzero morphisms. Our isogenies will be defined over k. If an isogeny is given by rational fractions of degree ℓ, it is called an ℓ-isogeny.

Complex multiplication

From now on, $k = \mathbb{F}_p$ is a prime finite field.
Let $E/\mathbb{F}_p$ be an ordinary elliptic curve.

- The ring $\mathrm{End}(E)$ is isomorphic to an order in a quadratic number field. The Frobenius endomorphism is a distinguished element in $\mathrm{End}(E)$.

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- The ring $\mathrm{End}(E)$ is isomorphic to an order in a quadratic number field. The Frobenius endomorphism is a distinguished element in $\mathrm{End}(E)$.
- Ideals of $\mathcal{O}$ modulo principal ideals form the *class group* of $\mathcal{O}$.

Isogenies of degree ℓ starting from E correspond to ideals in $\mathcal{O}$ of norm ℓ.
For example, in the generic case, there are either zero or two isogenies of degree ℓ with domain E.

Action of the class group

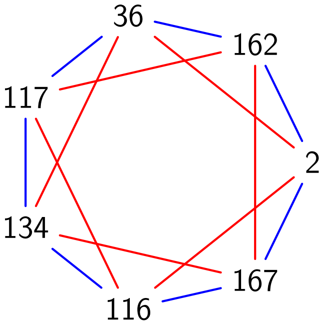
Proposition

- There is an action of the class group on a set of elliptic curves.
- Ideals of norm ℓ act as ℓ-isogenies.
- This action is simply transitive.

Therefore, in our setting, isogeny graphs are just Cayley graphs of a certain group.

Our isogeny graphs

Isogeny graph over $\mathbb{F}_{173}$ with isogenies of degree 3 (blue) and 7 (red):



This graph is *much* larger for cryptographic uses.

Representing isogenies

Let E/k be an elliptic curve, and $\ell \neq p$ be an odd prime. Giving the following is equivalent:

- An isogeny $E \to E'$ of degree ℓ
- Its kernel, which is a cyclic subgroup of E of order ℓ
- A polynomial of degree $\frac{\ell-1}{2}$ in x defining the kernel.

If we know this *kernel polynomial*, we can easily find E' using Vélu's formulas.

Representing ideals

We do *not* compute directly in the class group. Instead, we use the following representation of ideals:

If the ideal $\mathfrak{l}$ has norm ℓ, we have a natural surjection

$$\mathcal{O}/\ell\mathcal{O} \to \mathcal{O}/\mathfrak{l}\mathcal{O} \simeq \mathbb{Z}/\ell\mathbb{Z}.$$

The ideal ℓ is determined by the tuple (ℓ, v), where v is the image of the Frobenius under this surjection. We call v a *Frobenius eigenvalue*.

General algorithm

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The j-invariant we want is one of the two roots of a polynomial equation, called *modular equation*: $\Phi_\ell(j(E), Y) = 0$.

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Algorithm

Let E be a curve and (ℓ, v) be an ideal.

- compute and solve this equation: find j_1, j_2
- compute the kernel polynomial $K(x)$ of $E \to j_1$
- check if the Frobenius acts on it as scalar mult. by v:

$$(x^p, y^p) \stackrel{?}{=} [v] \cdot (x, y) \quad \text{mod } K(x) \text{ and curve equation.}$$

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Algorithm (Bostan–Morain–Salvy–Schost 2008)

- Compute power series solutions of this ODE up to a certain precision with a Newton iteration
- Recover $K(x)$ using the Berlekamp–Massey rational reconstruction algorithm.

Using Vélu's formulas

Problem

Given $E/\mathbb{F}_p$ and a prime $\ell \neq p$, how can we compute the curves linked to E by an ℓ-isogeny?

Finding roots of modular polynomials is costly : $\Phi_\ell(X, Y)$ has degree $\ell + 1$ in both variables.

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Another solution

Suppose that K is a subgroup of order ℓ in E whose points are defined over $\mathbb{F}_p$.

- Look for ℓ-torsion points over $\mathbb{F}_p$ to find K, using scalar multiplications
- Compute the curve E/K using Vélu's formulas.

The isogeny $E \to E/K$ has degree ℓ.

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- Using efficient arithmetic on curves is important (use other models than Weierstrass equations)
- Not every curve satisfies the previous conditions for many ℓ's and small d's: we have to look for adequate curves.
- In practice, we have to use both the general algorithm and Vélu's formulas.

What we would like to do

For the general method:

- Define elliptic curves over finite fields and general rings
- Define isogenies, scalar multiplication and isomorphisms
- Have a database of modular polynomials
- Find roots of polynomials over finite fields
- BMSS: ODEs in power series with Newton iterations and Berlekamp–Massey.

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Three ways to compute scalar multiplications

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Sol. 1 (Nemo)

```
E = Weierstrass(...)
Fext, _ = FiniteField(p, d, "alpha")
Eext = base_extend(E, Fext)
P = rand(Eext)
p^d * P
```

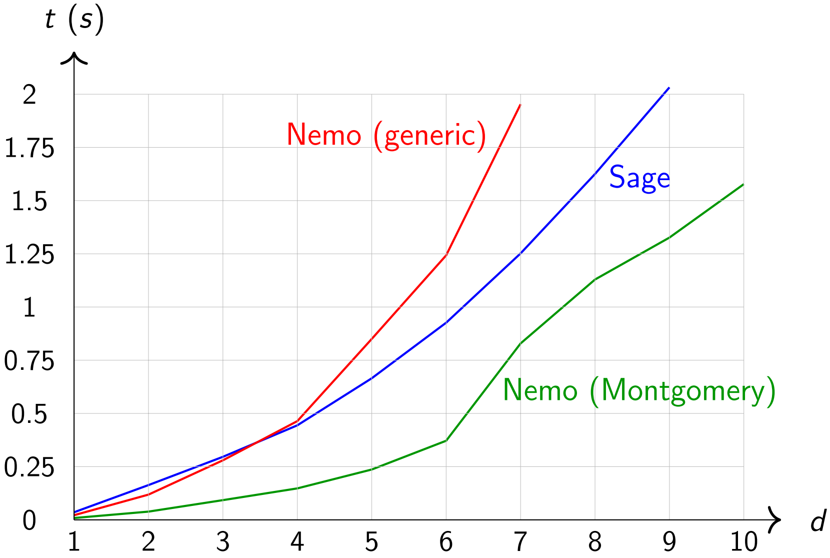
Sol. 2 (Nemo)

```
E = Montgomery(...)
Fext, _ = FiniteField(p, d, "alpha")
Eext = base_extend(E, Fext)
P = randXonly(Eext)
p^d * P
```

Sol. 3 (Sage)

```
E = EllipticCurve(...)
Fext = FiniteField(p**d, "alpha")
Eext = E.base_extend(Fext)
P = Eext.random_element()
C = p**d
C * P
```

Timing results



Further possible development

Around the previous algorithms:

- Call (system) PARI to compute the cardinality of curves over finite fields
- Have access to FLINT's root finding algorithms modulo p
- Have a decent system to handle field extensions
- Have p-adic numbers to compute isogenies in small characteristic?
- Connections with Hecke to be able to compute in endomorphism rings?

Further possible development

This module may also become useful to people learning about elliptic curves and elliptic curve cryptography:

- Implement other models for curves
- Add pairings
- ...

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- In order to use Vélu's formulas, we have to look for adequate curves, and this requires lots of computational power.
- With the best curve we found so far, aiming at 128-bit security, we reduced the computing time from 880 to 360 seconds. Better curves would bring further improvement.
- The EllipticCurves module is able to perform these computations.

Thank you!